

E. The quantities x and L have been treated as continuous functions of the time, but, as they represent numbers of persons, they can only take integral values, and must be discontinuous unless they are constant; in the same way, f and l can only vary by integral multiples of $1/P$. This is a serious drawback when the numbers are not large. In a small village, the average number of cases in the period between two epidemics may be only two or three, and, if these are removed by any chance, the whole course of events is entirely changed; there is no one to keep alive the infection, and a new generation may grow up entirely non-immune. Then the entry of a fresh source of infection has the same effect as if c rose suddenly from 0 to its normal value. In fact, in the case of small populations, other than the most probable values may often obtain, so that the whole of this paper is applicable to large numbers rather than to small ones.

The Initial Wave Resistance of a Moving Surface Pressure.

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1. The study of the water waves produced by the motion of an assigned pressure distribution over the surface has hitherto been limited to the steady state attained when the system has been moving with uniform velocity for a very long time. In his latest series of papers on water waves, Lord Kelvin* made an elaborate graphical and numerical study of cognate problems, and expressed the hope of applying his methods to calculate the initiation and continued growth of canal ship-waves due to the sudden commencement and continued application of a moving, steady surface pressure.

In the following paper, I have not attempted any analysis of the surface elevation itself, but I have proceeded directly to the calculation of the corresponding wave resistance. At present the wave resistance is known only for the steady state for certain localised pressure systems in uniform motion, and it seems desirable to attempt some estimate of the time taken to attain this state when we take into account the beginnings of the motion. One might examine the effect of initial acceleration, but I have limited the problem by considering only the case of a system which is suddenly established, and is at the same instant set in motion with uniform velocity.

* Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 456 (1906).

The work is arranged in the following order: a general expression for the wave resistance as a function of the time, an exact solution for a certain waveless system, a comparison of this solution and the group approximation, and an approximate solution for certain systems which leave regular waves in their rear.

2. Consider, first, the effect of a single impulse applied to the surface of deep water, with no initial displacement of the surface. Take the axis of y vertically upwards, the axis of x horizontal, and the origin in the undisturbed surface. If the impulse is given by $F(x)$, and if the Fourier method is applicable, the elevation at any time t is given by

$$-\pi g \rho y = \int_0^\infty \kappa V \sin(\kappa V t) d\kappa \int_{-\infty}^\infty F(\alpha) e^{i\kappa(x-\alpha)} d\alpha, \quad (1)$$

where $V = (g/\kappa)^{\frac{1}{2}}$, and the real part of the integral is to be taken. The effect of a pressure system, whether stationary or moving, can be obtained by integrating (1) suitably with respect to the time. For the pressure system may be considered as a succession of impulses; to each impulse there corresponds a fluid motion with definite velocity potential, and the velocity potential of the fluid motion at any instant is the sum of the velocity potentials due to all the previous impulses. Similarly, the corresponding surface elevations are simply superposed, and we obtain the required solution by an integration.

For a pressure system moving with uniform velocity c , we have to substitute $x+ct$ for x in (1) and then integrate with respect to t between the limits 0 and t . But the solution so obtained is indeterminate to a certain extent, for we can superpose on it any infinite train of waves of wave velocity c . The so-called practical solution is found by choosing the amplitude of this train so as to annul the main regular waves in front of the travelling system. The integrals are, in fact, indeterminate, and are evaluated by taking their principal value, in Cauchy's sense of the term. Another way of avoiding this difficulty is to introduce small frictional terms proportional to the velocity. The integrals are then determinate, though more complicated in form; however, the final results, after the analysis is completed, can be simplified by taking the frictional coefficient as small as we please. We shall use this method, and it is sufficient for our purpose to write, instead of (1),

$$-\pi g \rho y = \int_0^\infty e^{-\mu \kappa t} \kappa V \sin(\kappa V t) d\kappa \int_{-\infty}^\infty F(\alpha) e^{i\kappa(x-\alpha)} d\alpha, \quad (2)$$

where, ultimately, μ is to be considered small.*

* Compare Lamb, 'Hydrodynamics,' p. 406.

Consider, then, a pressure system

$$p = F(x), \quad (3)$$

which is suddenly established, and is at the same instant set in motion with uniform velocity c along the axis of x .

Putting $x = \varpi + ct$, the surface elevation at any time t after the start is given by

$$-\pi g \rho y = \int_0^t e^{-\mu c u} du \int_0^\infty \kappa V e^{i\kappa(\varpi + cu)} \sin(\kappa V u) d\kappa \int_{-\infty}^\infty F(\alpha) e^{-i\kappa \alpha} d\alpha. \quad (4)$$

For simplicity, we shall confine ourselves to pressure systems which are symmetrical with respect to the origin; so that

$$\phi(\kappa) = \int_{-\infty}^\infty F(\alpha) e^{-i\kappa \alpha} d\alpha = 2 \int_0^\infty F(\alpha) \cos \kappa \alpha d\alpha. \quad (5)$$

Also we shall use only localised distributions for which the integrals are finite and determinate; the systems will be finite and continuous and such that the integral pressure is finite, that is, the integral $\int_{-\infty}^\infty F(\alpha) d\alpha$ convergent. Carrying out the integration with respect to u , we obtain

$$\begin{aligned} -2\pi g \rho y &= \int_0^\infty \kappa V \phi(\kappa) e^{i\kappa \varpi} \left\{ \frac{1}{\kappa(V+c) + i\mu c} + \frac{1}{\kappa(V-c) - i\mu c} \right\} d\kappa \\ &- e^{-\mu c t} \int_0^\infty \kappa V \phi(\kappa) e^{i\kappa \varpi} \left\{ \frac{e^{i\kappa(V+c)t}}{\kappa(V+c) + i\mu c} + \frac{e^{-i\kappa(V-c)t}}{\kappa(V-c) - i\mu c} \right\} d\kappa. \end{aligned} \quad (6)$$

The first integral represents the steady state, while the second gives the deviation from it when we take into account the beginning of the motion.

3. From the first integral in (6) we have, with $\kappa_0 = g/c^2$,

$$-\pi g \rho y = \kappa_0 \lim_{\mu \rightarrow 0} \int_0^\infty \frac{\kappa \phi(\kappa) e^{i\kappa \varpi} d\kappa}{\kappa(\kappa_0 - \kappa) + \mu^2 - 2\mu \kappa i}. \quad (7)$$

The integral is to be evaluated first, before we make μ zero, otherwise it is indeterminate. The interpretation for certain types of localised pressure system is well known; in such cases the solution takes the form

$$\begin{aligned} y &= f(\varpi), \quad \varpi > 0, \\ y &= -\frac{2\kappa_0}{g\rho} \phi(\kappa_0) \sin \kappa_0 \varpi + f(-\varpi), \quad \varpi < 0. \end{aligned} \quad (8)$$

This solution represents an infinite train of regular waves in the rear of the moving system, together with a disturbance symmetrical fore and aft which becomes negligible at a distance depending upon the concentration and the velocity. For our present purpose, all the examples we use are included under the case

$$\phi(\kappa) = \kappa^n e^{-a\kappa}, \quad n > 0, \quad a > 0. \quad (9)$$

To verify the solution (8) in this case, regard κ in (7) as a complex variable $re^{i\theta}$.

For ϖ positive, integrate round a sector of radius R bounded by the lines $\theta = 0$ and $\theta = \beta$ ($0 < \beta < \frac{1}{2}\pi$). Under the specified conditions, it can be shown that the integral along the arc $r = R$ tends to zero as R is made infinite. In this way the integral (7) is transformed into an integral, along the line $\theta = \beta$, in which we can make μ zero.

For ϖ negative, integrate round a sector of radius R bounded by the lines $\theta = 0$ and $\theta = \beta$, with $-\tan^{-1}2\mu/\kappa_0 > \beta > -\frac{1}{2}\pi$. We get a similar result, except that the integrand has now a simple pole within the sector at the point $\kappa_0 - 2\mu i$ approximately. The residue at this pole gives the term in (8) which represents the regular train of waves in the rear of the system. It can also be verified that in this case y and $\partial y / \partial \varpi$ are finite and continuous throughout.

Returning to the general expression (6), the second integral represents the deviation from the steady state. It contains $\exp \{i\kappa(\varpi + ct)\}$ as a factor, and we see from its form that it represents the effect at time t of a certain initial distribution of velocity and displacement. To illustrate this point, consider a stationary pressure system which is suddenly established at a given instant and maintained constant. The effect is the same as if there had been in existence up to the given instant two equal and opposite systems with their ultimate static effect upon the water surface fully established, the negative system being then suddenly annulled. Thus the subsequent effect is the steady state of the positive system combined with the effect of an initial displacement equal to the steady state of an equal negative system. In the same way, for a pressure system which is suddenly established and started in uniform motion, the effect is the superposition of the steady state of this system and the disturbance due to initial conditions given by the steady state of an equal negative system in uniform motion. We shall find this principle of use in a later section.

4. The wave resistance R_1 in the steady state is usually obtained from energy principles applied to the regular waves. The front of the train advances with velocity c , while the rate of flow of energy across any fixed vertical plane in the rear is the corresponding group velocity $\frac{1}{2}c$; from the amplitude of the regular waves in (8), by equating the net rate of gain of fluid energy to $R_1 c$, it follows that

$$R_1 = \kappa_0^2 \{ \phi(\kappa_0) \}^2 / g\rho. \quad (10)$$

Some consideration is necessary before we can apply this method to the motion before the steady state has been attained.

Begin with a case in which there is no ambiguity, namely, when the waves are produced by a rigid body moving horizontally through the liquid. We can apply the general hydrodynamical principle that the rate of increase of total energy of the fluid is equal to the activity of the pressure taken over all the bounding surfaces. If we equate the rate of increase of energy to the product of a force R and the velocity of the rigid body, it follows that R is simply the total fluid pressure on the moving body resolved horizontally. This result can easily be verified by direct calculation for the steady state, whether the waves are produced by the motion of a rigid body or by the motion of an assigned surface pressure; in fact, the two cases are identical in the steady state, for we can imagine the surface pressure to be applied by a rigid cover which fits the water surface everywhere.

Consider now the problem before the steady state has been established. If the waves are caused by a moving rigid body, we can use either definition for the wave resistance; we can calculate it from the rate of increase of fluid energy or from the total horizontal pressure on the body. We are not discussing this case, simply because so far the analysis has proved too complicated to allow of suitable reduction. We replace this problem by that of the motion of an assigned surface pressure. Now we can calculate the rate of increase of the total energy of the fluid when the pressure system is in motion. But it would not be satisfactory to divide this quantity by the velocity of the pressure system and define the quotient as the wave resistance, for part of the increase of fluid energy is independent of the motion of the pressure system. For instance, if a stationary pressure system is suddenly established and maintained steady, the activity of the surface pressure is not zero immediately after the initial instant; there is a subsequent flow of energy, whose rate ultimately subsides to zero. From these considerations it seems that we should get results more comparable with the wave resistance of a rigid body by adopting the alternative method of calculation. In what follows we shall therefore calculate for any instant the total horizontal component of the surface pressure regarded as applied normally to the surface of the water; and we shall define this to be the wave resistance.

With the usual limitation that the slope of the surface is everywhere small, we have from this definition

$$R = - \int_{-\infty}^{\infty} F(\varpi) \frac{\partial \eta}{\partial \varpi} d\varpi. \quad (11)$$

We can verify that this gives the same result (10) for the steady state. For instance, taking the expressions in (8), the part which is symmetrical with respect to the origin gives no contribution to R , and we obtain

$$g\rho R_1 = 2 \int_{-\infty}^0 F(\varpi) \cdot \kappa_0^2 \phi(\kappa_0) \cos \kappa_0 \varpi d\varpi = \kappa_0^2 \{\phi(\kappa_0)\}^2,$$

or if we work directly from the integral (7), we have

$$\pi g\rho R_1 = \kappa_0 \lim_{\mu \rightarrow 0} \int_0^\infty \frac{i\kappa^2 \{\phi(\kappa)\}^2 d\kappa}{\kappa(\kappa_0 - \kappa) + \mu^2 - 2\mu\kappa i}, \quad (12)$$

where the real part is to be taken. Under the general conditions specified for $\phi(\kappa)$, or, in particular, for the case given in (9), it can be shown that this leads to the same expression (10). The wave resistance in general is the sum of two parts, the steady value R_1 , as given by (10), and the deviation R_2 .

Using the definition (11) with the second integral in (6), we find

$$-2\pi g\rho R_2 = \lim_{\mu \rightarrow 0} e^{-\mu ct} \int_0^\infty i\kappa^2 V \{\phi(\kappa)\}^2 \left[\frac{e^{-i\kappa(V-c)t}}{\kappa(V-c) - \mu ci} + \frac{e^{i\kappa(V+c)t}}{\kappa(V+c) + \mu ci} \right] d\kappa. \quad (13)$$

5. Consider first a special case in which the pressure system is such that there are no regular waves left in the rear, a type which Kelvin called a waveless system. It follows from (10), (12), and (13), that this is the case when the system is such that $\phi(\kappa)$ is of the form $(\kappa - \kappa_0)\psi(\kappa)$, where $\psi(\kappa)$ remains finite. We have then

$$\int_{-\infty}^\infty F(\varpi) \cos \kappa \varpi d\varpi = \phi(\kappa) = (\kappa - \kappa_0)\psi(\kappa). \quad (14)$$

If this system is made to travel with the velocity c , for which $2\pi/\kappa_0$ is the free wave-length, there will be no regular train of waves in the rear. The integrals (12) and (13) now remain finite and determinate with μ zero; we can thus simplify the expressions by making μ zero. The integral (12) vanishes, as does also the equivalent expression (10). Then, taking the real part of (13), we find for the total wave resistance of this system at any time t

$$-\pi g\rho R = \kappa_0^{\frac{3}{2}} \int_0^\infty \kappa(\kappa - \kappa_0) \{\psi(\kappa)\}^2 \times \{\kappa^{\frac{1}{2}} \sin \kappa Vt \cos \kappa ct - \kappa_0^{\frac{1}{2}} \cos \kappa Vt \sin \kappa ct\} d\kappa. \quad (15)$$

It is of interest to examine this solution when the integral can be evaluated exactly in finite terms. Burnside* suggested some years ago a method of building up exact solutions of certain wave problems, and similar forms have been analysed in detail by Kelvin, after obtaining the solutions by a different method. The cases in which we can carry out the integrations in (15) lead to similar functions; we obtain them by taking

$$\psi(\kappa) = \kappa^{\frac{1}{2}} e^{-r\kappa}, \quad r > 0. \quad (16)$$

* W. Burnside, 'Proc. Lond. Math. Soc.,' vol. 20, p. 31 (1888).

This case is the simplest of the type which allows of exact evaluation of (15), and for which the integral pressure is finite. To derive the corresponding pressure system, we make use of Euler's integrals of the form

$$\int_0^\infty \kappa^{n-1} e^{-\lambda \kappa \cos \alpha} \cos(\lambda \kappa \sin \alpha) d\kappa = \lambda^{-n} \Gamma(n) \cos n\alpha, \quad (17)$$

$$\lambda > 0, \quad n > 0, \quad -\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi.$$

Using the Fourier integral theorem, combined with (16) and (17), we find

$$\begin{aligned} \pi F(x) &= \int_0^\infty (\kappa - \kappa_0) \kappa^{\frac{1}{2}} e^{-\tau \kappa} \cos \kappa x d\kappa \\ &= \Gamma\left(\frac{9}{4}\right) (\tau^2 + x^2)^{-9/8} \cos\left(\frac{9}{4} \tan^{-1} x/\tau\right) - \kappa_0 \Gamma\left(\frac{5}{4}\right) (\tau^2 + x^2)^{-5/8} \cos\left(\frac{5}{4} \tan^{-1} x/\tau\right). \end{aligned} \quad (18)$$

The two terms of this expression are easily graphed when expressed in terms of the angle $\tan^{-1}(x/\tau)$; two numerical cases are shown later.

We can now find the resistance R for the system (18), travelling with velocity $c = \sqrt{g/\kappa_0}$. Substituting (17) in (15), and writing $\kappa = u^2$, $g^{\frac{1}{2}}t = q$, $ct = p$, we have

$$\begin{aligned} -\frac{1}{2}\pi g \rho R &= -\kappa_0 \int_0^\infty (u^6 - \kappa_0 u^4) e^{-2ru^2} \sin pu^2 \cos qu du \\ &\quad + \kappa_0^{\frac{3}{2}} \int_0^\infty (u^7 - \kappa_0 u^5) e^{-2ru^2} \cos pu^2 \sin qu du. \end{aligned} \quad (19)$$

The integrals involved can all be derived, by differentiation with respect to the parameters, from

$$\int_0^\infty e^{-(\rho - ip)u^2} \cos qu du = \frac{1}{2} [\pi/(\rho - ip)]^{\frac{1}{2}} e^{-q^2/4(\rho - ip)}. \quad (20)$$

Carrying out these operations, we obtain finally

$$\begin{aligned} -\pi^{1/2} g \rho R &= \kappa_0^{1/2} (4r^2 + c^2 t^2)^{-5/4} e^{-g^{\frac{1}{2}} t^2 r/2 (4r^2 + c^2 t^2)} \\ &\quad \times [\kappa_0^{\frac{1}{2}} \{ -\frac{1}{8} A \sin(\frac{7}{2}\theta - \phi) + \frac{4}{16} q^2 A^2 \sin(\frac{9}{2}\theta - \phi) - \frac{1}{3 \cdot 2} q^4 A^3 \sin(\frac{11}{2}\theta - \phi) \\ &\quad + \frac{1}{6 \cdot 4} q^6 A^4 \sin(\frac{13}{2}\theta - \phi) \} + \kappa_0^{3/2} \{ \frac{3}{4} \sin(\frac{5}{2}\theta - \phi) - \frac{3}{4} q^2 A \sin(\frac{7}{2}\theta - \phi) \\ &\quad + \frac{1}{16} q^4 A^2 \sin(\frac{9}{2}\theta - \phi) \} + \frac{1}{16 \cdot 5} q A^2 \cos(\frac{9}{2}\theta - \phi) - \frac{1}{3 \cdot 2} q^3 A^3 \cos(\frac{11}{2}\theta - \phi) \\ &\quad + \frac{2}{6 \cdot 4} q^5 A^4 \cos(\frac{13}{2}\theta - \phi) - \frac{1}{1 \cdot 2 \cdot 8} q^7 A^5 \cos(\frac{15}{2}\theta - \phi) \\ &\quad + \kappa_0 \{ -\frac{1}{8} q A \cos(\frac{7}{2}\theta - \phi) + \frac{5}{8} q^3 A^2 \cos(\frac{9}{2}\theta - \phi) - \frac{1}{3 \cdot 2} q^5 A^3 \cos(\frac{11}{2}\theta - \phi) \}], \end{aligned} \quad (21)$$

where

$$q = g^{\frac{1}{2}} t; \quad A = (4r^2 + c^2 t^2)^{-\frac{1}{2}}; \quad \theta = \tan^{-1}(ct/2r); \quad \phi = gct^3/4(4r^2 + c^2 t^2).$$

6. Before working out numerical examples, it is convenient to record the asymptotic expansion suitable for large values of $ct/2r$. From (21), by

writing $\theta = \frac{1}{2}\pi - \tan^{-1}(2r/ct)$, and expanding the various terms, we get, up to and including terms in $(2r/ct)^{3/2}$,

$$-\pi^{1/2}g\rho R = \frac{9}{128}g^4c^{-17/2}t^{-1/2}e^{-\beta} \cos(gt/4c + \frac{1}{4}\pi) + \frac{1}{64}g^3c^{-15/2}t^{-3/2}\{37 + \beta(18\beta - 75)\}e^{-\beta} \cos(gt/4c - \frac{1}{4}\pi), \quad (22)$$

where $\lambda_0 = 2\pi c^2/g$ and $\beta = \pi r/\lambda_0$. If the pressure system has moved through n wave-lengths, we have $ct = n\lambda_0$, and the ratio of the amplitudes of the two terms in (22) is

$$\frac{1}{9\pi n} \left\{ 37 + \frac{\pi r}{\lambda_0} \left(\frac{18\pi r}{\lambda_0} - 75 \right) \right\}, \quad (23)$$

an expression which gives some estimate of the approximation obtained by using only the first term of (22). It depends not only upon the distance travelled, but also upon the ratio of the effective breadth of the system to the free wave-length for the assigned velocity.

Compare this approximation with that obtained by applying Kelvin's group method directly to the integral expression for the wave resistance.

Under certain conditions,* an approximate value of an integral of the form

$$\int_a^b F(\kappa) e^{i f(\kappa)} d\kappa$$

is given by

$$\frac{\sqrt{\pi} F(\alpha)}{\sqrt{|\frac{1}{2}f''(\alpha)|}} e^{i\{f(\alpha) \pm \pi/4\}}, \quad (24)$$

the upper or lower sign being taken in the exponential according as $f''(\alpha)$ is positive or negative, and α being a root of $f'(\alpha) = 0$. It is assumed that the circular function in the integral goes through a large number of periods within the range of integration, while $F(\kappa)$ changes comparatively slowly; in addition, the quotient $f'''(\alpha)/\{f''(\alpha)\}^{3/2}$ must be small.

Apply this to the form for F given in (13). The second term within the square brackets contributes nothing to the approximation; from the first term we have, with $ct = n\lambda_0$,

$$f(\kappa) = -\kappa(V-c)t = -g^{\frac{1}{2}}t\kappa^{\frac{3}{2}} + ct\kappa.$$

Hence

$$\alpha = g/4c^2; \quad f''(\alpha) = 2c^3t/g; \quad f'''(\alpha)/\{f''(\alpha)\}^{3/2} = 3/\sqrt{(\pi n)}.$$

From (13) and (24) the group value of R is

$$R = -\frac{1}{2\pi g\rho} \lim_{\mu \rightarrow 0} e^{-\mu ct} \sqrt{\left(\frac{\pi g}{c^3t}\right) \frac{ig^{1/2}\alpha^{3/2}\{\phi(\alpha)\}^2}{g^{1/2}\alpha^{1/2} - c\alpha - i\mu c}} e^{i(gt/4c - \pi/4)}. \quad (25)$$

Taking the real part of this expression and putting μ zero, we obtain

$$R = -\frac{g^{1/2}\{\phi(g/4c^2)\}^2}{4\pi^{1/2}\rho c^{7/2}t^{1/2}} \cos(gt/4c + \frac{1}{4}\pi). \quad (26)$$

* Lamb, 'Hydrodynamics,' p. 385.

It should be noted that for a pressure system which leaves regular waves in its rear, we cannot take (26) as an approximation for the limiting value of (13) when $\mu \rightarrow 0$, except under certain further limitations. For the present this difficulty does not arise, as we are considering a waveless system with $\phi(\kappa)$ of the form $(\kappa - \kappa_0)\psi(\kappa)$; we have seen that in this case the integrals (12) and (13) remain finite and determinate with μ zero.

In particular, for the forms (16) and (18), the group formula (26) reduces to

$$-\pi^{1/2}g\rho R = \frac{9}{128}g^4c^{-17/2}t^{-1/2}e^{-\pi r/\lambda_0}\cos(gt/4c + \frac{1}{4}\pi), \quad (27)$$

which, from (22), agrees with the first term in the asymptotic expansion of the exact solution for this case.

Instead of expressing R as a function of the time t , we can use the distance travelled, or again the number n of free wave-lengths λ_0 through which the system has moved; in the last case the circular function in (27) becomes $\cos \frac{1}{4}(2n+1)\pi$. The form of (27) agrees with the definition of the wave resistance as the resolved total pressure. For after a sufficient time, the surface in the neighbourhood of the moving origin consists chiefly of the simple waves whose group velocity is the velocity c of the pressure system; thus the wave-length there is $4\lambda_0$.

7. Consider now two numerical examples of the exact solution (21) with different values of the ratio r/λ_0 .

In the first place, we shall adopt units used by Kelvin, for comparison and for simplicity of calculation.

Case i: $g = 4$; $r = 1$; $\lambda_0 = 2$; $\kappa_0 = \pi$; $c = 2/\sqrt{\pi}$.

From (18) the pressure system $F(x)$ can be obtained by graphing

$$4\pi \cos^{5/4}\theta \cos \frac{5}{4}\theta - 5 \cos^{9/4}\theta \cos \frac{9}{4}\theta,$$

where $\theta = \tan^{-1}(x/r)$. The graph is shown in curve (1) of fig. 1; the curve has maxima near $x = \pm 0.2$, though they are almost inappreciable on the diagram.

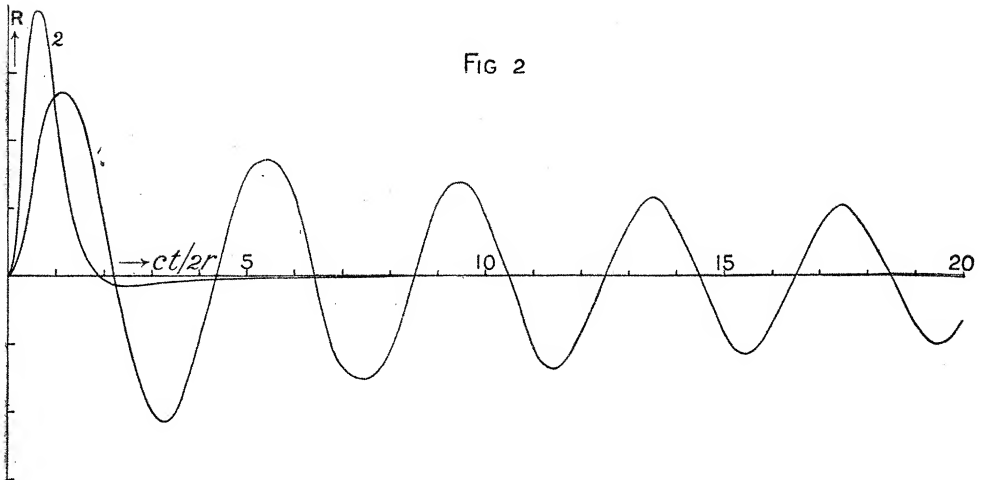
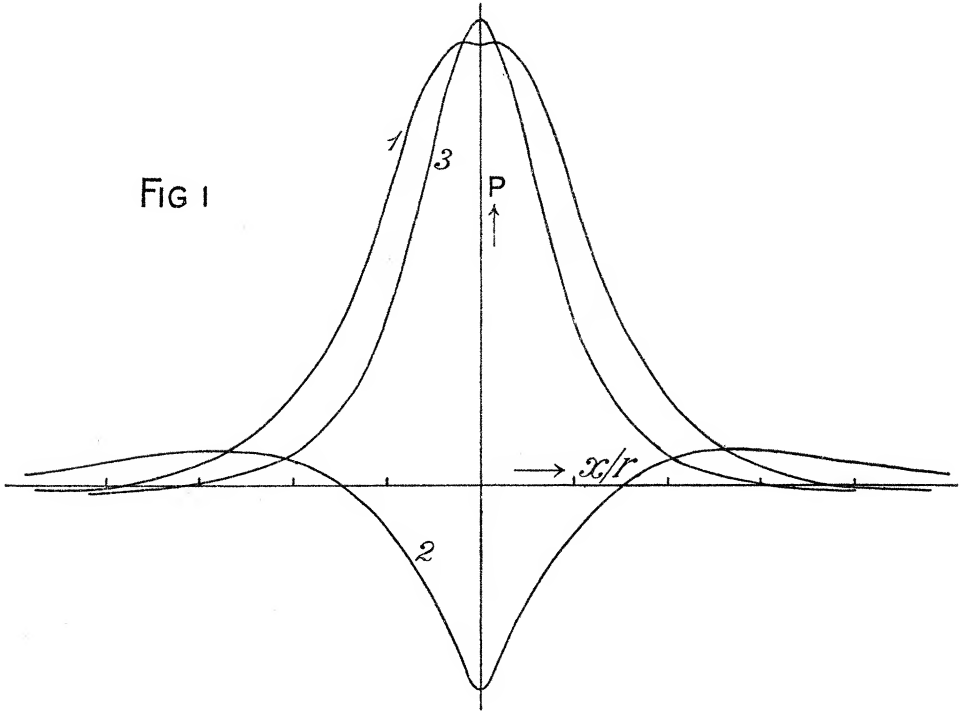
It is convenient to graph the resistance curve upon a base $\xi = ct/2r$; in this particular case ξ is also the number of wave-lengths λ_0 through which the system has moved. The angles of the formula (21) are now

$$\theta = \tan^{-1}\xi; \quad \phi = \pi\xi^3/2(1+\xi^2).$$

It is unnecessary to repeat the expression (21) with these values; each of the 14 terms can be easily calculated for any given value of ξ . The results are shown in curve (1) of fig. 2; to obtain the curve 15 points were calculated by the formula (21).

The wave resistance decreases ultimately to zero, as it should for a waveless

system, but it approaches the steady state very slowly. This is explained when we examine the graph of the pressure system in this case. The



waveless character is due to the mutual interference effects produced by the peaks of the pressure graph, and fig. 1 shows how inappreciable the peak

are in this case. Hence the slowness with which the steady state is attained and the probable lack of stability of the steady state.

To compare the group approximation with the exact solution, we have from (27)

$$-\pi^{1/2}g\rho R = 9 \times 2^{-15/2}\pi^4 n^{-1/2} e^{-\pi/2} \cos \frac{1}{4}(2n+1)\pi. \quad (28)$$

The following is a comparison of the values of $10^2\pi^{1/2}g\rho R$, as given by (28) and the exact formula (21):

n .	Group.	Exact.
9	-23.73	-26.45
16	+17.77	+16.04
25	-14.21	-14.93
100	+ 7.11	+ 6.94

Case ii.—As a second numerical example, we take one which might correspond more to practical conditions, in that the pressure system is similar to that associated with the motion of a ship model in an experimental tank. Using foot-second units, we take

$$g = 32; \quad r = 2; \quad c = 20; \quad \kappa_0 = 0.08; \quad \lambda_0 = 25\pi.$$

The pressure system is graphed in curve (2) of fig. 1, from the expression $8 \cos^{5/4} \theta \cos \frac{5}{4} \theta - 125 \cos^{9/4} \theta \cos \frac{9}{4} \theta$. We notice the contrast between this and the previous case. We should now expect the steady state to be attained quickly and to be much more stable. This is brought out very clearly by the resistance curve, which has been graphed from (21), and is shown in curve (2) of fig. 2; after the initial peak, the subsequent oscillations can scarcely be shown on the scale of the diagram.

A comparison of the exact formula and the group approximation gives similar results to the previous case, for in both the numerical value of the ratio (23) is of the order $1/n$, in spite of the difference in the values of r/λ_0 for the two cases.

It should be remarked that the two cases cannot be compared as regards absolute values from the curves shown, because the scale for the ordinates has been chosen arbitrarily in each case. The maximum value of R , that is, the value at the prominent peak on curve (2), is given by $g\rho R = 7 \times 10^{-3}$. We can obtain some idea of the magnitude by the following comparison: We have chosen the pressure system so that it is waveless at a particular velocity, namely, 20 feet per second. Now, imagine the same system to be

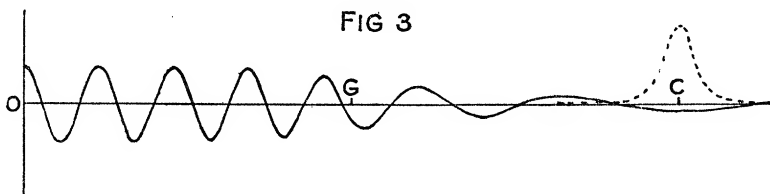
driven at any other steady velocity ; it will have a steady resistance, which we can calculate from the formula (10); in this case it is

$$g\rho R = \kappa^{5/2}(\kappa - 0.08)^2 e^{-4\kappa}, \quad \kappa = g/V^2. \quad (29)$$

This steady wave resistance has a maximum at a velocity of about 5.25 ft./sec., and the value of $g\rho R$ is then 16.4×10^{-3} . Hence the maximum resistance due to the sudden starting of the system at its waveless speed is about one-half the maximum steady resistance at any uniform speed.

8. We have been able to obtain an exact solution for a special type of waveless system ; we leave this now to consider more generally a symmetrical localised pressure system, which is suddenly established and set in motion.

We have seen that the surface elevation at any time is found by superposing the steady state of the system and the effect due to initial conditions given by the steady state of an equal negative system in uniform motion. Apply this to a case in which the steady state consists of an infinite train of regular waves in rear of the system, together with a localised displacement symmetrical with respect to the moving origin. Let O be the fixed origin and starting point, and C the position at time t . The deviation from the ultimate steady state consists of the effect due to a certain initial distribution of displacement and velocity localised round O, together with the subsequent state of a semi-infinite train of regular waves, which at the initial instant had a definite front at the point O. We may describe the latter part in general terms as a regular train with a front, more or less definite according to the time, at a point G corresponding to the group velocity, and in advance of G a disturbance which may be called the forerunner. If OC is sufficiently large, and if we require the surface elevation only at points sufficiently far in advance of G, the forerunner is given with considerable accuracy by Kelvin's group method of approximation. The argument is represented diagrammatically in fig. 3, the continuous line showing the elevation and the dotted line the travelling pressure system.



The wave resistance being defined as the total horizontal component of the pressure system, we divide it into two parts. The first part is the final steady value $\kappa_0^2 \{\phi(\kappa_0)\}^2 / g\rho$ as given in (10), and the second is the deviation given by the integral in (13). The latter represents the resolved pressure

system as if the surface elevation were that due to the stoppage of a negative system, as represented in fig. 3.

For a concentrated pressure system, the value of the integral (13) will be given approximately by the Kelvin group method, if the time is sufficiently large; that is, if C is sufficiently far in advance of G for us to neglect the contribution of the applied pressure acting on the surface to the rear of G .

Without attempting to specify these conditions more precisely, we shall apply the method to the type of system used in the previous sections; from the previous exact solution we have been able to estimate somewhat the degree of accuracy of the group approximation.

The group value of (13) is given in (26). Hence the wave resistance, for sufficiently large values of t , is given by

$$R_1 = \frac{g}{\rho c^4} \left\{ \phi \left(\frac{g}{c^2} \right) \right\}^2 - \frac{g^{1/2} \{ \phi(g/4c^2) \}^2}{4\pi^{1/2} \rho c^{7/2} t^{1/2}} \cos(gt/4c + \frac{1}{4}\pi). \quad (30)$$

9. Apply this to the pressure distribution

$$\pi F(x) = \Gamma\left(\frac{9}{4}\right) (r^2 + x^2)^{-9/8} \cos \left\{ \frac{9}{4} \tan^{-1}(x/r) \right\}, \quad (31)$$

for which $\phi(\kappa) = \kappa^{5/4} e^{-r\kappa}$, with $\kappa = g/c^2$. The graph of this distribution is shown in curve (3) of fig. 1.

We have

$$\rho R = \frac{g^{7/2}}{c^9} e^{-2gr/c^2} - \frac{g^3}{128\pi^{1/2} c^{17/2} t^{1/2}} e^{-gr/2c^2} \cos(gt/4c + \frac{1}{4}\pi). \quad (32)$$

The value of R oscillates about the final steady value. The relative deviation is given by the ratio of the two terms, namely,

$$2^{-15/2} \pi^{-1} n^{-1/2} e^{3\pi r/\lambda_0} \cos \frac{1}{4} (2n+1)\pi,$$

where λ_0 is the wave-length of the regular train and $ct = \lambda n_0$. We may obtain numerical values by using the two cases of the previous sections.

For Case i we have $r = 1$, $\lambda_0 = 2$, and we find the following comparison between R_1 , the final steady resistance, and R_2 , the deviation given by the second term of (32):—

n .	R_2/R_1 .
9	+0·046
16	−0·035
25	+0·027
100	−0·014

Hence, after the system has moved through nine wave-lengths, the deviation is less than 5 per cent.

In Case ii, $r = 2$ and $\lambda_0 = 25\pi$. We find that when $n = 9$, the deviation is already less than 0.06 per cent.

10. Consider now a simpler type of localised pressure distribution, namely,

$$\pi F(x) = r/(\tau^2 + x^2). \quad (33)$$

This type leads to a steady wave resistance whose variation with the velocity is more like that of a ship model. We have $\phi(\kappa) = e^{-r\kappa}$, and (30) gives

$$\rho R = \frac{g}{c^4} e^{-2gr/c^2} - \frac{g^{1/2}}{4\pi^{1/2} c^{7/2} t^{1/2}} e^{-gr/2c^2} \cos(gt/4c + \frac{1}{4}\pi). \quad (34)$$

The relative deviation is now 32 times as large as in the previous case, since

$$\frac{R_2}{R_1} = \frac{e^{3\pi r/\lambda_0}}{2^{5/2} \pi n^{1/2}} \cos \frac{1}{4}(2n+1)\pi.$$

With $r = 1$, $\lambda_0 = 2$, the value of R_2/R_1 is about 0.5 for $n = 100$. We should have to take n of the order 10,000 before bringing the deviation from the steady* value below 5 per cent.

On the other hand, with $r = 2$, $\lambda_0 = 25\pi$, the deviation is under 2 per cent. when $n = 9$, or at about 35 seconds after the beginning of the motion; it is less than 2 per cent. when $n = 4$, or after a travel of rather more than 300 feet.

11. The waves produced by the horizontal motion of a circular cylinder of small radius travelling at a considerable depth h below the surface may be compared with those produced by the surface pressure

$$\pi F(x) = A c^2 (h^2 - x^2)/(h^2 + x^2)^2. \quad (35)$$

We assume that the intensity of the system is proportional to the square of the velocity. It appears that the steady wave resistance is then the same function of the velocity as in the motion of the cylinder;* for we have

$$\phi(\kappa) = A c^2 \kappa e^{-\kappa h},$$

and hence

$$\rho R = \frac{A^2 g^3}{c^4} e^{-2gh/c^2} - \frac{A^2 g^{5/2}}{64 \pi^{1/2} c^{7/2} t^{1/2}} e^{-gh/2c^2} \cos(gt/4c + \frac{1}{4}\pi). \quad (36)$$

As a numerical example, take the case when the velocity is such that the steady resistance R_1 has its maximum value; that is, when $c^2 = gh$. Then we have

$$\frac{R_2}{R_1} = \frac{c^{3/2}}{2^{13/2} \pi n^{1/2}} \cos \frac{1}{4}(2n+1)\pi. \quad (37)$$

The value of the ratio means a deviation from the steady value of about 0.8 per cent. when $n = 3\frac{1}{2}$, that is, when the system has travelled through a distance $7\pi h$.

* Lamb, 'Hydrodynamics,' p. 408.